

# Random walks in a queueing network environment

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**Abstract.** We propose a class of models of random walks in a random environment where an exact solution can be given for a stationary distribution. The environment is cast in terms of a Jackson/Gordon-Newell network although alternative interpretations are possible. The main tool is the detailed balance equations. The difference with earlier works is that the position of the random walk influences the transition intensities of the network environment and vice versa, creating strong correlations. The form of the stationary distribution is closely related to the well-known product-formula.

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## 1 Introduction

This paper introduces exactly solvable reversible models of a random walk interacting with a random environment of a queueing-network type. The environment stems from a symmetric Jackson network or its closed version (a Gordon–Newell network); cf. [10, 11, 7]. The walking particles can be interpreted as distinguished customers (DCs). Depending on the site of the network a DC is in, he/she may like staying in the site if the queue size is large (or small) and in turn encourages more

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(or less) tasks to come to this site. Further development is where DCs interact with each other: here we consider the case of a symmetric simple exclusion; the zero-range model is also included. By analyzing the detailed balance equations (DBEs), the equilibrium distribution is derived, closely related to the product-form distribution; cf. [1, 18, 20]. Our approach can be considered as a further development of earlier papers [5, 23] where possible applications have been outlined, in the area of communication networks.

Other areas of applications may cover problems of random trapping/localization and condensation; cf. [2, 6, 8, 22], and the bibliography therein.

An immediate problem would be to identify a Lyapunov function assessing the speed of convergence to the stationary (equilibrium) distribution under sub-criticality conditions. Cf. [17, 9] and references therein.

A possible step would be to introduce similar models in quasi-reversible setting [13]. In this direction, it would be interesting to treat a wider class of service rules for tasks and the DC. Viz., a processor-sharing discipline seems a natural choice, involving a branching formalism; see, e.g., [19].

Our models also show some potential in the direction of Markov processes with local interaction. An immediate goal can be to develop models in an infinite-volume configuration space [21], [3], [4].

## 2 Description of a basic model

### 2.1 A symmetric open Jackson network

As a model for an environment, we take a symmetric and homogeneous Jackson job-shop network; see [10, 11]. The model is defined by the following ingredients.

- (a) A finite collection  $\Lambda$  of sites with a single-server system assigned to each  $k \in \Lambda$ .
- (b) Two positive numbers:  $\lambda > 0$ , the intensity of an exogenous input flow to a given site, and  $\mu > 0$ , the intensity of the flow from a given site out of the network.
- (c) A non-negative symmetric matrix of transmission intensities  $B = (\beta_{ik}, i, k \in \Lambda)$ :

$$\beta_{ik} = \beta_{ki}, \quad \beta_{ik} \geq 0, \quad \beta_{ii} = 0. \quad (2.1)$$

The value  $\lambda$  gives the rate of independent Poisson processes of exogenous tasks arriving at sites in  $\Lambda$  and  $\bar{\mu}_i = \mu + \bar{\beta}_i$  is the rate of servicing the queue at site  $i \in \Lambda$  where  $\bar{\beta}_i = \sum_{k \in \Lambda} \beta_{ik}$ . After completing service at site  $i$ , a task leaves the network with probability  $\mu/\bar{\mu}_i$  and jumps to site  $k$  with probability  $\beta_{ik}/\bar{\mu}_i$ . Condition  $\beta_{ii} = 0$  is used for convenience. In contrast, the symmetry property  $\beta_{ik} = \beta_{ki}$  in (2.1) is essential but rather restrictive and hopefully could be weakened in future.

The above description gives rise to a continuous-time Markov process (MP) with states  $\underline{n} = (n_i, i \in \Lambda) \in \mathbb{Z}_+^\Lambda$  where  $n_i \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ . The generator matrix  $\mathbf{Q} = (Q(\underline{n}, \underline{n}'))$  of the process has non-zero entries corresponding to the following transitions:

$$\begin{aligned} Q(\underline{n}, \underline{n} + \underline{e}^i) &= \lambda && \text{an exogenous arrival of a task at site } i, \\ Q(\underline{n}, \underline{n} - \underline{e}^i) &= \mu \mathbf{1}(n_i \geq 1) && \text{a task exits from site } i \text{ out of the network,} \\ Q(\underline{n}, \underline{n} + \underline{e}^{i \rightarrow k}) &= \beta_{ik} \mathbf{1}(n_i \geq 1) && \text{a task jumps from site } i \text{ to } k. \end{aligned} \quad (2.2)$$

Here  $\underline{e}^i = (e_l^i, l \in \Lambda) \in \mathbb{Z}_+^\Lambda$  has  $e_l^i = \delta_{il}$ ,  $\underline{e}^{i \rightarrow k}$  denotes the difference  $\underline{e}^k - \underline{e}^i$ , and  $\mathbf{1}$  stands for the indicator.

Assuming the sub-criticality condition  $\frac{\lambda}{\mu} < 1$ , the invariant measure  $\pi$  is the product of geometric distributions with parameter  $\lambda/\mu$ . Here the probability  $\pi(\underline{n})$  of having  $n_i$  tasks at sites  $i \in \Lambda$  is of the form

$$\pi(\underline{n}) = \left(1 - \frac{\lambda}{\mu}\right)^{\#\Lambda} \prod_{i \in \Lambda} \left(\frac{\lambda}{\mu}\right)^{n_i}, \quad \underline{n} = (n_i) \in \mathbb{Z}_+^\Lambda. \quad (2.3)$$

In fact, assuming that matrix  $B$  is irreducible (i.e.,  $B^t$  has strictly positive entries for some  $t \in \mathbb{Z}_+$ ), the process is positive recurrent (PR).

Eqn (2.3) follows from the DBEs for probabilities  $\pi(\underline{n})$ ,  $\underline{n} \in \mathbb{Z}_+^\Lambda$ :

$$\begin{aligned} \pi(\underline{n})Q(\underline{n}, \underline{n} + \underline{e}^i) &= \pi(\underline{n} + \underline{e}^i)Q(\underline{n} + \underline{e}^i, \underline{n}), \quad \pi(\underline{n})Q(\underline{n}, \underline{n} - \underline{e}^j) = \pi(\underline{n} - \underline{e}^j)Q(\underline{n} - \underline{e}^j, \underline{n}), \quad n_j \geq 1, \\ \pi(\underline{n})Q(\underline{n}, \underline{n} + \underline{e}^{k \rightarrow l}) &= \pi(\underline{n} + \underline{e}^{k \rightarrow l})Q(\underline{n} + \underline{e}^{k \rightarrow l}, \underline{n}), \quad n_k \geq 1, \end{aligned}$$

which are easy to check. Note that the probabilities  $\pi(\underline{n})$  in (2.3) do not refer to matrix  $B$ .

In the whole paper,  $\Lambda$  stands for a finite set, and sites of  $\Lambda$  are marked by  $i, j, k, l, p, q, r, s$  and  $j'$ .

## 2.2 Random walk in a Jackson environment

We now give the description of the model with interaction. A new ingredient is the presence of a random particle (a distinguished customer (DC)) walking over set  $\Lambda$ . Parameters of the Jackson network are changed only for a site where the DC is located; we call it a *loaded* site and denote by  $j$ . A site  $i \neq j$  (free of a DC) is called *unloaded*. In general,  $i, k$  and  $l$  denote sites in  $\Lambda$  which may be loaded or unloaded.

In addition to  $\lambda, \mu$  and  $B = (\beta_{ik})$  (see items (b) and (c) in Section 2.1), we need more ingredients and rules. The queue of tasks at the loaded site  $j$  has an exogenous arrival intensity  $e^\varphi \lambda$  where  $\varphi \in \mathbb{R}$ , while for the remaining sites the intensity remains  $\lambda$ . The exit flow intensity from all sites of the network equals  $\mu$  as before. The intensities  $\beta_{ik}$  for  $i \neq j \neq k$  (task jumps from one unloaded site to another) are as before (and satisfy (2.1)). Next, we introduce symmetric matrices  $\Theta = (\theta_{ik}, i, k \in \Lambda)$  and  $T = (\tau_{ik}, i, k \in \Lambda)$  where

$$\theta_{ik} = \theta_{ki}, \quad \theta_{ik} \geq 0, \quad \theta_{ii} = 0; \quad \tau_{ik} = \tau_{ki}, \quad \tau_{ik} \geq 0, \quad \tau_{ii} = 0. \quad (2.4)$$

(Again, condition  $\theta_{ii} = \tau_{ii} = 0$  is used for convenience.) For the loaded site  $j$ , the intensity of the task flow from  $j$  to  $k \neq j$  equals  $e^\varphi \theta_{jk}$ . For an unloaded site  $k \neq j$ , the intensity of the task flow from  $k$  to  $j$  equals  $\theta_{kj}$ . Finally, the intensity of a leap of the DC from a loaded site  $j$  where there are  $n_j$  tasks to a site  $j' \neq j$  is taken to be  $e^{-\varphi n_j} \tau_{jj'}$ . As with intensities  $\beta_{ik}$ , symmetry equations  $\theta_{ik} = \theta_{ki}$  and  $\tau_{ik} = \tau_{ki}$  are essential (in modified forms, they reappear in the rest of the paper), and it would be interesting to replace these conditions with weaker ones.

Matrices  $\Theta$  and  $T$  and parameter  $\varphi \in \mathbb{R}$  are additional ingredients of the model. (In future we also refer to  $B$ ,  $\Theta$  and  $T$  as arrays, or collections of intensities.) The state of the emerging MP is a pair  $(j, \underline{n})$  where  $j \in \Lambda$  and  $\underline{n} = (n_i, i \in \Lambda) \in \mathbb{Z}_+^\Lambda$ . As was said, the entry  $j$  indicates the loaded site, and  $n_i$ , as before, gives the number of tasks at site  $i \in \Lambda$ . In accordance with the above description, the generator  $\mathbf{R} = \left\{ R[(j, \underline{n}); (j', \underline{n}')] \right\}$  has the following entries (here and below, zero-rate transitions are not shown):

$$\begin{aligned}
R[(j, \underline{n}); (j, \underline{n} + \underline{e}^k)] &= \lambda e^{\varphi \delta_{jk}} && \text{exogenous arrival of a task,} \\
R[(j, \underline{n}); (j, \underline{n} - \underline{e}^k)] &= \mu \mathbf{1}(n_k \geq 1) && \text{exit of a task out of the network,} \\
R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{k \rightarrow l})] &= \beta_{kl} \mathbf{1}(n_k \geq 1) && \text{jump of a task, for } k \neq j \neq l, \\
R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{j \rightarrow l})] &= \theta_{jl} \mathbf{1}(n_j \geq 1) && \text{a task jumps from the loaded site, for } l \neq j, \\
R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{k \rightarrow j})] &= \theta_{kj} e^\varphi \mathbf{1}(n_k \geq 1) && \text{a task jumps to the loaded site, for } k \neq j, \\
R[(j, \underline{n}); (j', \underline{n})] &= e^{-\varphi n_j} \tau_{jj'} && \text{leap of a DC, from } j \text{ to } j' \neq j.
\end{aligned} \tag{2.5}$$

Pictorially, the DC is served or provides service affecting task arrivals (both exogenous and intrinsic) at the loaded site, after which it moves to another site. Conversely, the DC departure from the loaded site is influenced by the number of tasks accumulated in the queue. If  $\varphi < 0$  then the presence of the DC suppresses task arrivals, whereas  $\varphi > 0$  means the opposite. Likewise, the intensity of a DC leap increases with  $n_j$  for  $\varphi < 0$  and decreases for  $\varphi > 0$ .

### 3 An exact solution for a single DC

#### 3.1 The basic case

In this sub-section we assume that rate collections  $B$ ,  $\Theta$  and  $T$  satisfy (2.1), (2.4) and are irreducible (i.e., matrices  $B^u$ ,  $\Theta^u$  and  $T^u$  have strictly positive entries for some positive integer  $u$ ).

**Theorem 3.1.** *Assume that the sub-criticality condition holds true:  $\frac{\lambda}{\mu}, \frac{\lambda e^\varphi}{\mu} < 1$ . Then the MP on  $\Lambda \times \mathbb{Z}_+^\Lambda$  with generator  $\mathbf{R}$  is positive recurrent and reversible (PRR). The stationary probability (SP)  $\pi(j, \underline{n})$ ,  $j \in \Lambda$ ,  $\underline{n} = (n_i) \in \mathbb{Z}_+^\Lambda$ , of locating the DC at site  $j$  and having  $n_i$  tasks in sites of  $\Lambda$  is of the*

form

$$\pi(j, \underline{n}) = (\#\Lambda)^{-1} \left(1 - \frac{\lambda e^\varphi}{\mu}\right) \left(1 - \frac{\lambda}{\mu}\right)^{\#\Lambda - 1} \left(\frac{\lambda}{\mu}\right)^{\sum_{i \in \Lambda} n_i} e^{\varphi n_j}. \quad (3.1)$$

*Proof.* Probabilities  $\pi(j, \underline{n})$  from (3.1) satisfy the following DBEs:

$$\begin{aligned} \pi(j, \underline{n}) R[(j, \underline{n}); (j, \underline{n} + \underline{e}^k)] &= \pi(j, \underline{n} + \underline{e}^k) R[(j, \underline{n} + \underline{e}^k); (j, \underline{n})], \quad k \neq j, \\ \pi(j, \underline{n}) R[(j, \underline{n}); (j, \underline{n} + \underline{e}^j)] &= \pi(j, \underline{n} + \underline{e}^j) R[(j, \underline{n} + \underline{e}^j); (j, \underline{n})], \\ \pi(j, \underline{n}) R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{k \rightarrow l})] &= \pi(j, \underline{n} + \underline{e}^{k \rightarrow l}) R[(j, \underline{n} + \underline{e}^{k \rightarrow l}); (j, \underline{n})], \quad k \neq j \neq l \neq k, \quad n_k > 0, \\ \pi(j, \underline{n}) R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{j \rightarrow k})] &= \pi(j, \underline{n} + \underline{e}^{j \rightarrow k}) R[(j, \underline{n} + \underline{e}^{j \rightarrow k}); (j, \underline{n})], \quad j \neq k, \quad n_j > 0, \\ \pi(j, \underline{n}) R[(j, \underline{n}); (j', \underline{n})] &= \pi(j', \underline{n}) R[(j', \underline{n}); (j, \underline{n})], \quad j \neq j'. \end{aligned} \quad (3.2)$$

In fact, substituting (2.5), omitting constant factors and canceling the common term  $\left(\frac{\lambda}{\mu}\right)^{\sum_i n_i}$  yields the identities:

$$\begin{aligned} e^{\varphi n_j} \lambda &= \left(\frac{\lambda}{\mu}\right) e^{\varphi n_j} \mu, \quad k \neq j, \quad e^{\varphi n_j} \lambda e^\varphi = \left(\frac{\lambda}{\mu}\right) e^{\varphi(n_j+1)} \mu, \\ e^{\varphi n_j} \beta_{kl} &= e^{\varphi n_j} \beta_{lk}, \quad k \neq j \neq l \neq k, \quad n_k > 0, \\ e^{\varphi n_j} \theta_{jk} &= e^{\varphi(n_j-1)} \theta_{kj} e^\varphi, \quad j \neq k, \quad n_j > 0, \quad e^{\varphi n_j} e^{-\varphi n_j} \tau_{jj'} = e^{\varphi n_{j'}} e^{-\varphi n_{j'}} \tau_{j'j}, \quad j \neq j'. \end{aligned}$$

Finally, under the irreducibility assumption, the process is PR.  $\square$

Note that the SP distribution  $\pi$  in (3.1) does not involve B,  $\Theta$ , T. The same pattern will be observed in the generalizations of the basic model below.

### 3.2 A direct generalization

In this sub-section, the rates  $R[(j, \underline{n}), (j', \underline{n}')] are as follows (cf. (2.5)):$

$$\begin{aligned} R[(j, \underline{n}); (j, \underline{n} + \underline{e}^k)] &= \lambda_k(n_k; j), \quad k \neq j, \quad R[(j, \underline{n}); (j, \underline{n} + \underline{e}^j)] = \lambda_j(n_j; j) \gamma_j(n_j), \\ R[(j, \underline{n}); (j, \underline{n} - \underline{e}^k)] &= \mu_k(n_k, k) \mathbf{1}(n_k \geq 1), \\ R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{k \rightarrow l})] &= \beta_{kl}(n_k, n_l; j) \mathbf{1}(n_k \geq 1), \quad k \neq j \neq l \neq k, \\ R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{j \rightarrow l})] &= \theta_{jl}(n_j, n_l) \mathbf{1}(n_j \geq 1), \quad l \neq j, \\ R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{k \rightarrow j})] &= \theta_{kj}(n_k, n_j) \gamma_j(n_j) \mathbf{1}(n_k \geq 1), \quad k \neq j, \\ R[(j, \underline{n}); (j', \underline{n})] &= [\bar{\gamma}_j(n_j)]^{-1} \tau_{jj'}(\underline{n}), \quad j \neq j'. \end{aligned} \quad (3.3)$$

Let us comments on the form of these rates. As one can see, we take into account the numbers of tasks in related sites and the position of the DC. To start with, we deal in (3.3) with values  $\lambda_i(n; j)$  and  $\mu_i(n; j)$ ,  $n \in \mathbb{Z}_+$ ,  $i, j \in \Lambda$ . For  $i \neq j$ , values  $\lambda_i(n; j)$ ,  $\mu_i(n; j)$  give the intensities of exogenous arrival and exit of tasks, depending on  $i$ , the site location,  $n(= n_i)$ , the current number of tasks at

the site, and on  $j$ , the current loaded site. For  $i = j$ ,  $\mu_j(n; j)$  yields an exit intensity from a loaded site while  $\lambda_j(n; j)$  represents a nominal arrival intensity which will be modified via a gauge function  $\gamma_j(n)$ . Examples are

$$\lambda_i(n; j) = \lambda^U \mathbf{1}(n < C), \quad \mu_i(n; j) = \mu^U \min[n, K], \quad j \neq i,$$

$$\lambda_j(n; j) = \lambda^L \mathbf{1}(n < C), \quad \mu_j(n; j) = \mu^L \min[n, K],$$

where  $\lambda^U$ ,  $\lambda^L$ ,  $\mu^U$ ,  $\mu^L$ ,  $C$  and  $K$  are positive constants (which can be made varying the site). It means that the arrival at a given (unloaded) site is blocked if the number of tasks reaches  $C$ , and the pre-exit service is done by a  $K$ -server device, with intensities depending on the site.

We assume for simplicity that

$$\sup [\lambda_i(n; j) : n \geq 0, i; j \in \Lambda] < +\infty, \quad \inf [\mu_i(n; j) : n \geq 1, i; j \in \Lambda] > 0, \quad (3.4)$$

and that  $\lambda_i(n; j) > 0$  if  $\lambda_i(n+1; j) > 0$ . Through the whole paper, we also set:

$$\bar{\lambda}_i(0; j) = \bar{\mu}_i(0; j) = 1, \quad \bar{\lambda}_i(n; j) = \prod_{0 \leq m < n} \lambda_i(m; j), \quad \bar{\mu}_i(n; j) = \prod_{1 \leq m \leq n} \mu_i(m; j), \quad n > 0. \quad (3.5)$$

We also work with values  $\gamma_i(n) \geq 0$ ,  $i \in \Lambda$ ,  $n \in \mathbb{Z}_+$ , assuming that  $\gamma_i(0) = 1$  and  $\gamma_i(n) > 0$  if  $\gamma_i(n+1) > 0$ . These values are used to modify intensities of task arrival and jump at loaded site  $j$ . Viz.,

$$\gamma_j(0) = 1, \quad \gamma_j(n) = \frac{e^{\varphi(j)}}{n} \quad \text{or} \quad \gamma_j(n) = ne^{\varphi(j)}, \quad n \geq 1,$$

where  $\varphi(j)$  is a given real parameter depending upon  $j$ . Next, we let, again throughout the paper,

$$\bar{\gamma}_i(0) = 1, \quad \bar{\gamma}_i(n) = \prod_{0 \leq m < n} \gamma_i(m), \quad n \geq 1. \quad (3.6)$$

Further, the intensities  $\beta_{ik}$  depend on  $j$  and  $n_i$ ,  $n_k$ , and we write  $B(\underline{n}; j) = (\beta_{ik}(n_i, n_k; j))$ . Consequently, we modify symmetry assumptions in (2.1): for  $i \neq k$ ,

$$\frac{\lambda_i(n_i - 1; j)}{\mu_i(n_i; j)} \beta_{ik}(n_i, n_k; j) = \frac{\lambda_k(n_k; j)}{\mu_k(n_k + 1; j)} \beta_{ki}(n_i + 1, n_k - 1; j), \quad (3.7)$$

and set  $\beta_{ii}(n_i, n_i; j) = 0$ . Similar conditions are imposed on array  $\Theta(\underline{n}; j) = (\theta_{ik}(n_i, n_k; j))$ : for  $i \neq k$ ,

$$\frac{\lambda_i(n_i - 1; j)}{\mu_i(n_i; j)} \theta_{ik}(n_i, n_k; j) = \frac{\lambda_k(n_k; j)}{\mu_k(n_k + 1; j)} \theta_{ki}(n_k + 1, n_i - 1; j), \quad (3.8)$$

and  $\theta_{ii}(n_i, n_i; j) = 0$ .

Finally, intensities  $\tau_{ik}$  depend upon  $\underline{n}$  yielding an array  $T(\underline{n}) = (\tau_{ik}(\underline{n}), i, k \in \Lambda)$ . In this section we assume that  $\tau_{ii}(\underline{n}) = 0$  and for  $i \neq k$  and  $\underline{n} = (n_q, q \in \Lambda) \in \mathbb{Z}_+^\Lambda$ ,

$$\tau_{ik}(\underline{n}) \prod_{q \in \Lambda} \frac{\bar{\lambda}_q(n_q; i)}{\bar{\mu}_q(n_q; i)} = \tau_{ki}(\underline{n}) \prod_{q \in \Lambda} \frac{\bar{\lambda}_q(n_q; k)}{\bar{\mu}_q(n_q; k)}. \quad (3.9)$$

We continue referring to (3.7) – (3.9) as symmetry conditions; one of primary future tasks should be their replacement with less restrictive assumptions.

In Theorem 3.2 we assume conditions (3.7) – (3.9) (these conditions will be recast in Section 4 in a more general situation) and suppose that arrays  $B(\underline{n}; j)$ ,  $\Theta(\underline{n}; j)$  and  $T(\underline{n})$  have off-diagonal entries  $> 0$ . (We keep referring to this property as irreducibility.) The sub-criticality condition reads

$$U_{ij} := \sum_{n \in \mathbb{Z}_+} \frac{\bar{\lambda}_i(n; j)}{\bar{\mu}_i(n; j)} < +\infty, \quad L_j := \sum_{n \in \mathbb{Z}_+} \frac{\bar{\lambda}_j(n; j) \bar{\gamma}_j(n)}{\bar{\mu}_j(n; j)} < \infty, \quad \forall \quad i, j \in \Lambda, \quad i \neq j. \quad (3.10)$$

**Theorem 3.2.** *Under (3.10), the MP on  $\Lambda \times \mathbb{Z}_+^\Lambda$  with generator  $\mathbf{R} = \left( R[(j, \underline{n}); (j', \underline{n}')] \right)$  as in (3.3) is PRR. The SP  $\pi(j, \underline{n})$ , of having the DC at site  $j$  and  $n_q$  tasks at  $q \in \Lambda$ , is of the form*

$$\pi(j, \underline{n}) = \frac{1}{\Xi_\Lambda} \prod_{q \in \Lambda} \frac{\bar{\lambda}_q(n_q; j)}{\bar{\mu}_q(n_q; j)} \bar{\gamma}_j(n_j), \quad j \in \Lambda, \underline{n} = (n_q) \in \mathbb{Z}_+^\Lambda, \quad \text{with} \quad \Xi_\Lambda = \sum_{j \in \Lambda} L_j \prod_{i \in \Lambda \setminus \{j\}} U_{ij}. \quad (3.11)$$

Here  $\Xi_\Lambda$  is the partition function of the model.

*Proof.* Probabilities  $\pi(j, \underline{n})$  from (3.11) and rates  $R[(j, \underline{n}), (j', \underline{n}')] from (3.3) satisfy the DBEs (3.2). In fact, after omitting the factor  $\frac{1}{\Xi_\Lambda}$  and canceling common terms in  $\prod_{i \in \Lambda} \frac{\bar{\lambda}_i(n_i; j)}{\bar{\mu}_i(n_i; j)}$ , the DBEs are again verified with the help of symmetry conditions: for  $j', j, k, l \in \Lambda$ , with  $k \neq j \neq l \neq k, j \neq j'$ ,$

$$\begin{aligned} \bar{\gamma}_j(n_j) \lambda_k(n_k; j) &= \frac{\lambda_k(n_k; j) \bar{\gamma}_j(n_j)}{\mu_k(n_k + 1; j)} \mu_k(n_k + 1; j), \\ \bar{\gamma}_j(n_j) \lambda_j(n_j; j) \gamma_j(n_j) &= \frac{\lambda_j(n_j; j) \bar{\gamma}_j(n_j + 1)}{\mu_j(n_j + 1; j)} \mu_j(n_j + 1; j), \\ \frac{\lambda_k(n_k - 1; j)}{\mu_k(n_k; j)} \bar{\gamma}_j(n_j) \beta_{kl}(n_k, n_l; j) &= \frac{\lambda_l(n_l; j) \bar{\gamma}_j(n_j)}{\mu_l(n_l + 1; j)} \beta_{lk}(n_l + 1, n_k - 1; j), \quad n_k \geq 1, \\ \frac{\lambda_j(n_j - 1; j) \bar{\gamma}_j(n_j)}{\mu_j(n_j; j)} \theta_{jk}(n_j, n_k; j) &= \frac{\lambda_k(n_k; j) \bar{\gamma}_j(n_j - 1)}{\mu_k(n_k + 1; j)} \theta_{kj}(n_k + 1, n_j - 1; j) \gamma_j(n_j - 1), \quad n_j \geq 1, \\ \bar{\gamma}_j(n_j) \bar{\gamma}_j(n_j)^{-1} \tau_{jj'}(\underline{n}) &= \bar{\gamma}_{j'}(n_{j'}) \bar{\gamma}_{j'}(n_{j'})^{-1} \tau_{j'j}(\underline{n}). \end{aligned}$$

As before, the process is PR under the irreducibility assumption.  $\square$

### 3.3 A closed-network version

This version arises when we keep fixed the number of tasks in the network. Correspondingly, we drop the two first lines in (3.3):

$$\begin{aligned} R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{k \rightarrow l})] &= \beta_{kl}(n_k, n_l; j) \mathbf{1}(n_k \geq 1), \quad k \neq j \neq l, \\ R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{j \rightarrow l})] &= \theta_{jl}(n_j, n_l) \mathbf{1}(n_j \geq 1), \\ R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{k \rightarrow j})] &= \theta_{kj}(n_k, n_j) \gamma_j(n_j) \mathbf{1}(n_k \geq 1), \\ R[(j, \underline{n}); (j', \underline{n})] &= [\bar{\gamma}_j(n_j)]^{-1} \tau_{jj'}(\underline{n}), \quad j \neq j'. \end{aligned} \quad (3.12)$$

In Theorem 3.3 we assume a modification of condition (3.9)

$$\tau_{jj'}(\underline{n}) = \tau_{j'j}(\underline{n}). \quad (3.13)$$

**Theorem 3.3.** Fix  $N$ , the number of tasks in the network. Given  $\underline{n} \in \mathbb{Z}_+^\Lambda$ , set:  $|\underline{n}| = \sum_{s \in \Lambda} n_s$ . The MP on  $\{(j, \underline{n}) \in \Lambda \times \mathbb{Z}_+^\Lambda : |\underline{n}| = N\}$  with generator  $\mathbf{R} = \left( R[(j, \underline{n}); (j', \underline{n}')] \right)$  as in (3.12) is PRR. The SPs  $\pi(j, \underline{n})$  take the form

$$\pi(j, \underline{n}) = \frac{\mathbf{1}(|\underline{n}| = N)}{\Xi_{N, \Lambda}} \bar{\gamma}_j(n_j) \quad \text{where} \quad \Xi_{N, \Lambda} = \sum_{\substack{\underline{n} = (n_s) \in \mathbb{Z}_+^\Lambda \\ |\underline{n}| = N}} \sum_{l \in \Lambda} \bar{\gamma}_l(n_l).$$

*Proof.* Again, we use the DBEs verified with the help of the corresponding symmetry conditions:

$$\begin{aligned} \pi(j, \underline{n}) R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{i \rightarrow k})] &= \pi(j, \underline{n} + \underline{e}^{i \rightarrow k}) R[(j, \underline{n} + \underline{e}^{i \rightarrow k}); (j, \underline{n})], \quad i \neq j \neq k, n_i > 0, \\ \pi(j, \underline{n}) R[(j, \underline{n}); (j, \underline{n} + \underline{e}^{j \rightarrow k})] &= \pi(j, \underline{n} + \underline{e}^{j \rightarrow k}) R[(j, \underline{n} + \underline{e}^{j \rightarrow k}); (j, \underline{n})], \quad j \neq k, n_j > 0, \\ \pi(j, \underline{n}) R[(j, \underline{n}); (j', \underline{n})] &= \pi(j', \underline{n}) R[(j', \underline{n}); (j, \underline{n})], \quad j \neq j'. \quad \square \end{aligned}$$

## 4 Simple exclusion in a Jackson-type environment

### 4.1 A closed-open network

The simple exclusion model was introduced in [21] (where the corresponding term has been coined). The model was extensively studied thereafter: cf. [14], [15], [16]. Here the state of the MP is a pair  $(\underline{y}, \underline{n})$  where  $\underline{y} = (y_s, s \in \Lambda) \in \{0, 1\}^\Lambda$  and  $\underline{n} = (n_i, i \in \Lambda) \in \mathbb{Z}_+^\Lambda$ . Let us set:  $|\underline{y}| = \sum_{s \in \Lambda} y_s$ . We also write  $j \in \underline{y}$  when  $y_j = 1$  and  $j \notin \underline{y}$  when  $y_j = 0$ . In the case of a closed-open network, the sum  $M = |\underline{y}|$  remains constant. The rates (now denoted by  $R[(\underline{y}, \underline{n}); (\underline{y}', \underline{n}')] )$  are specified as

$$\begin{aligned} R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^p)] &= \lambda_p(n_p; \underline{y}) [\gamma_p(n_p)]^{y_p}, \quad R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} - \underline{e}^p)] = \mu_p(n_p; \underline{y}) \mathbf{1}(n_p \geq 1), \\ R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{k \rightarrow l})] &= \beta_{kl}(n_k, n_l; \underline{y}), \quad R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{i \rightarrow j})] = \epsilon_{ij}(n_i, n_j; \underline{y}), \\ R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{i \rightarrow l})] &= \theta_{il}(n_i, n_l; \underline{y}), \quad R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{k \rightarrow j})] = \theta_{kj}(n_k, n_j; \underline{y}) \gamma_j(n_j), \\ R[(\underline{y}, \underline{n}); (\underline{y} + \underline{e}^{j \rightarrow j'}, \underline{n})] &= [\bar{\gamma}_j(n_j)]^{-1} \tau_{jj'}(\underline{n}; \underline{y}), \end{aligned} \quad (4.1)$$



assuming (i)  $i \neq j$ ,  $i, j \in \underline{y}$ ,  $k \neq l$ ,  $k, l, j' \notin \underline{y}$ , and (ii)  $n_i, n_k \geq 1$ . Here we deal with intensities  $\lambda_\bullet(\cdot; \underline{y})$  and  $\mu_\bullet(\cdot; \underline{y})$  depending on  $\underline{y}$ . Such a generalization is extended to arrays  $B(\underline{n}; \underline{y}) = (\beta_{kl}(n_k, n_l; \underline{y}))$ ,  $\Theta(\underline{n}; \underline{y}) = (\theta_{kl}(n_k, n_l; \underline{y}))$  and  $T(\underline{n}; \underline{y}) = (\tau_{jj'}(\underline{n}; \underline{y}))$ . We assume symmetry conditions similar to (3.7), (3.8) and (3.9): for  $j \in \underline{y}$ , and  $k \neq l$ ,  $k, l, j' \notin \underline{y}$ ,

$$\begin{aligned} \frac{\lambda_k(n_k - 1; \underline{y})}{\mu_k(n_k; \underline{y})} \beta_{kl}(n_k, n_l; \underline{y}) &= \frac{\lambda_l(n_l; \underline{y})}{\mu_l(n_l + 1; \underline{y})} \beta_{lk}(n_l + 1, n_k - 1; \underline{y}), \quad n_k \geq 1, \\ \frac{\lambda_j(n_j - 1; \underline{y})}{\mu_j(n_j; \underline{y})} \theta_{jk}(n_j, n_k; \underline{y}) &= \frac{\lambda_k(n_k; \underline{y})}{\mu_k(n_k + 1; \underline{y})} \theta_{kj}(n_k + 1, n_j - 1; \underline{y}), \quad n_j \geq 1, \end{aligned} \quad (4.2)$$

and – see Eqn (3.5) –

$$\tau_{jj'}(\underline{n}; \underline{y}) \prod_{q \in \Lambda} \frac{\bar{\lambda}_q(n_q; \underline{y})}{\bar{\mu}_q(n_q; \underline{y})} = \tau_{j'j}(\underline{n}; \underline{y} + \underline{e}^{j \rightarrow j'}) \prod_{q \in \Lambda} \frac{\bar{\lambda}_q(n_q; \underline{y} + \underline{e}^{j \rightarrow j'})}{\bar{\mu}_q(n_q; \underline{y} + \underline{e}^{j \rightarrow j'})}. \quad (4.3)$$

We also have a new collection of jump rates  $E(\underline{n}; \underline{y}) = (\epsilon_{ij}(n_i, n_j; \underline{y}))$  satisfying the symmetry property: for  $i \neq j$ ,  $i, j \in \underline{y}$  and  $n_i \geq 1$ ,

$$\frac{\lambda_i(n_i - 1; \underline{y}) \gamma_i(n_i - 1)}{\mu_i(n_i; \underline{y})} \epsilon_{ij}(n_i, n_j; \underline{y}) = \frac{\lambda_j(n_j; \underline{y}) \gamma_j(n_j)}{\mu_j(n_j + 1; \underline{y})} \epsilon_{ji}(n_j + 1, n_i - 1; \underline{y}). \quad (4.4)$$

Until the end of Section 4 we work with irreducible collections  $B(\underline{n}; \underline{y})$ ,  $\Theta(\underline{n}; \underline{y})$ ,  $E(\underline{n}; \underline{y})$  and  $T(\underline{n}; \underline{y})$ . Furthermore, assumptions (4.2) – (4.4) are adopted in sub-Sections 4.1 and 4.2. The interpretation is that we have a 1-0 configuration  $\underline{y}$  of loaded sites occupied by DCs, with a total number of DCs  $M$ ; each of them influences task arrivals and task jumps as indicated, independently for different sites. In addition, each DC can jump from a loaded to an unloaded site, again independently.

With  $\bar{\gamma}_j(n)$  as in (3.6), the SPs  $\pi(\underline{y}, \underline{n})$  read

$$\pi(\underline{y}, \underline{n}) = \frac{\mathbf{1}(|\underline{y}| = M)}{\Xi_{\Lambda, M}} \prod_{q \in \Lambda} \frac{\bar{\lambda}_q(n_q; \underline{y})}{\bar{\mu}_q(n_q; \underline{y})} \prod_{j \in \Lambda: y_j = 1} \bar{\gamma}_j(n_j) \quad (4.5)$$

with partition function  $\Xi_{\Lambda, M} = \sum_{\substack{\underline{y}=(y_s) \in \{0,1\}^\Lambda: \\ |\underline{y}|=M}} \prod_{r: y_r=1} L_r(\underline{y}) \prod_{l: y_l=0} U_l(\underline{y})$ ,  $M \leq \# \Lambda$ , and

$$U_l(\underline{y}) = \sum_{n \in \mathbb{Z}_+} \frac{\bar{\lambda}_l(n; \underline{y})}{\bar{\mu}_l(n; \underline{y})}, \quad L_r(\underline{y}) = \sum_{n \in \mathbb{Z}_+} \frac{\bar{\lambda}_r(n; \underline{y}) \bar{\gamma}_r(n)}{\bar{\mu}_r(n; \underline{y})}. \quad (4.6)$$

The sub-criticality conditions emerging from (4.6) are:  $\forall \underline{y} \in \{0, 1\}^\Lambda$  with  $|\underline{y}| = M$ ,

$$U_l(\underline{y}) < +\infty, \quad L_j(\underline{y}) < +\infty, \quad \forall l, j \in \Lambda \text{ with } y_l = 0 \text{ and } y_j = 1. \quad (4.7)$$

**Theorem 4.1.** *The MP with generator  $\mathbf{R} = \left( R[(\underline{y}, \underline{n}); (\underline{y}', \underline{n}')] \right)$  as in (4.1) on state space  $\left\{ (\underline{y}, \underline{n}) \in \{0, 1\}^\Lambda \times \mathbb{Z}_+^\Lambda : |\underline{y}| = M \right\}$  is PRR. The SPs  $\pi(\underline{y}, \underline{n})$  are given by (4.5).*

*Proof.* As before, the proof is based on DBEs. These are now as follows: for  $j, j', k, l \in \Lambda$ , with  $k \neq l$ ,

$$\begin{aligned} \pi(\underline{y}, \underline{n}) R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^k)] &= \pi(\underline{y}, \underline{n} + \underline{e}^k) R[(\underline{y}, \underline{n} + \underline{e}^k); (\underline{y}, \underline{n})], \quad k \notin \underline{y}, \\ \pi(\underline{y}, \underline{n}) R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^j)] &= \pi(\underline{y}, \underline{n} + \underline{e}^j) R[(\underline{y}, \underline{n} + \underline{e}^j); (\underline{y}, \underline{n})], \quad j \in \underline{y}, \\ \pi(\underline{y}, \underline{n}) R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{k \rightarrow l})] &= \pi(\underline{y}, \underline{n} + \underline{e}^{k \rightarrow l}) R[(\underline{y}, \underline{n} + \underline{e}^{k \rightarrow l}); (\underline{y}, \underline{n})], \quad y_k = y_l, \quad n_k \geq 1, \\ \pi(\underline{y}, \underline{n}) R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{j \rightarrow k})] &= \pi(\underline{y}, \underline{n} + \underline{e}^{j \rightarrow k}) R[(\underline{y}, \underline{n} + \underline{e}^{j \rightarrow k}); (\underline{y}, \underline{n})], \quad j \in \underline{y}, \quad k \notin \underline{y}, \quad n_j \geq 1, \\ \pi(\underline{y}, \underline{n}) R[(\underline{y}, \underline{n}); (\underline{y} + \underline{e}^{j \rightarrow j'}, \underline{n})] &= \pi(\underline{y} + \underline{e}^{j \rightarrow j'}, \underline{n}) R[(\underline{y} + \underline{e}^{j \rightarrow j'}, \underline{n}); (\underline{y}, \underline{n})], \quad j \in \underline{y}, \quad j' \notin \underline{y}. \end{aligned} \quad (4.8)$$

The verification is still direct. For definiteness, we show the equation emerging in the third line of (4.8), when  $k, l \in \underline{y}$  (other cases have been effectively considered earlier). Upon omitting the factor  $\frac{1}{\Xi_\Lambda}$  and canceling common terms in the products  $\prod_{j \in \Lambda: y_j=1} \bar{\gamma}_j(n_j)$  and  $\prod_{q \in \Lambda} \frac{\bar{\lambda}_q(n_q; j)}{\bar{\mu}_q(n_q; j)}$ , this equation becomes (4.4).  $\square$

## 4.2 An open-open network

Now the rates from (4.1) are complemented with

$$\begin{aligned} R[(\underline{y}, \underline{n}); (\underline{y} + \underline{e}^k, \underline{n})] &= \xi_k \bar{\gamma}_k(n_k) \mathbf{1}(y_k = 0), \quad \text{arrival of a DC at site } k, \\ R[(\underline{y}, \underline{n}); (\underline{y} - \underline{e}^i, \underline{n})] &= \eta_i \mathbf{1}(y_i = 1), \quad \text{exit of a DC from site } i. \end{aligned} \quad (4.9)$$

Here  $\xi_k > 0$  and  $\eta_i > 0$  do not depend on  $\underline{n}$  or  $\underline{y}$ . (This assumption can be generalized but certain independence should be maintained.) Further, in this sub-section we assume Eqn (4.2). In addition, we assume here that,  $\forall \underline{n} \in \mathbb{Z}_+^\Lambda$ ,

$$\text{the product } V(\underline{n}) = \prod_{q \in \Lambda} \frac{\bar{\lambda}_q(n_q; \underline{y})}{\bar{\mu}_q(n_q; \underline{y})} \text{ does not depend on configuration } \underline{y} \in \{0, 1\}^\Lambda. \quad (4.10)$$

Accordingly, Eqn (4.3) has to be replaced with

$$\tau_{jj'}(\underline{n}; \underline{y}) \frac{\xi_j}{\eta_j} = \tau_{j'j}(\underline{n}; \underline{y} + \underline{e}^{j \rightarrow j'}) \frac{\xi_{j'}}{\eta_{j'}}, \quad j \in \underline{y}, \quad j' \notin \underline{y}. \quad (4.11)$$

The SPs  $\pi(\underline{y}, \underline{n})$  become

$$\pi(\underline{y}, \underline{n}) = \frac{V(\underline{n})}{\Xi_\Lambda} \prod_{j \in \Lambda: y_j=1} \frac{\xi_j \bar{\gamma}_j(n_j)}{\eta_j}. \quad (4.12)$$

Here  $\Xi_\Lambda = \sum_{\underline{y} \in \{0,1\}^\Lambda} \sum_{\underline{n} \in \mathbb{Z}_+^\Lambda} V(\underline{n}) \prod_{j \in \Lambda: y_j=1} \frac{\xi_j \bar{\gamma}_j(n_j)}{\eta_j} = \sum_{\underline{y} \in \{0,1\}^\Lambda} \prod_{r: y_r=1} \frac{\xi_r L_r(\underline{y})}{\eta_r} \prod_{l \in \Lambda: y_l=0} U_l(\underline{y})$ , and  $L_r(\underline{y})$  and  $U_l(\underline{y})$  are as in (4.6). The sub-criticality condition reads  $\Xi_\Lambda < \infty$ , or

$$U_l(\underline{y}) < +\infty, L_r(\underline{y}) < +\infty, \quad \forall \underline{y} \in \{0,1\}^\Lambda \text{ and } l, r \in \Lambda \text{ with } y_l = 0 \text{ and } y_r = 1, \quad (4.13)$$

and can be treated as a slight modification of (4.7).

**Theorem 4.2.** *Under (4.13), the MP on  $\{0,1\}^\Lambda \times \mathbb{Z}_+^\Lambda$  with generator  $\mathbf{R} = \left( R[(\underline{y}, \underline{n}); (\underline{y}', \underline{n}')] \right)$  as in (4.1), (4.8) is PRR. The SPs  $\pi(\underline{y}, \underline{n})$  are given by (4.12).*

*Proof.* As before, one checks the DBEs (4.8) completed with

$$\pi(\underline{y}, \underline{n}) R[(\underline{y}, \underline{n}); (\underline{y} + \underline{e}^k, \underline{n})] = \pi(\underline{y} + \underline{e}^k, \underline{n}) R[(\underline{y} + \underline{e}^k, \underline{n}); (\underline{y}, \underline{n})], \quad k \notin \underline{y}. \quad (4.14)$$

In view of (4.10), the latter holds true.  $\square$

### 4.3 A closed-closed network

Here we keep  $|\underline{y}|$  and  $|\underline{n}|$  fixed:  $|\underline{y}| = M$ ,  $|\underline{n}| = N$ . The rates are as in Eqn (4.1), with the top three lines discarded. The following conditions are assumed: for  $i \neq j$ ,  $i, j \in \underline{y}$  and  $k \neq l$ ,  $k, l, j' \notin \underline{y}$ ,

$$\begin{aligned} \beta_{kl}(n_k, n_l; \underline{y}) &= \beta_{lk}(n_l + 1, n_k - 1; \underline{y}), \quad \theta_{jk}(n_j, n_k; \underline{y}) = \theta_{kj}(n_k + 1, n_j - 1; \underline{y}), \quad n_k, n_j \geq 1, \\ \epsilon_{ij}(n_i, n_j; \underline{y}) \gamma_i(n_i - 1) &= \gamma_j(n_j) \epsilon_{ji}(n_j + 1, n_i - 1; \underline{y}), \quad n_i \geq 1, \end{aligned} \quad (4.15)$$

and

$$\tau_{jj'}(\underline{n}; \underline{y}) = \tau_{j'j}(\underline{n}; \underline{y} - \underline{e}^j + \underline{e}^{j'}), \quad (4.16)$$

which can be viewed as a modification of (4.2) – (4.4) and (3.13).

The SP distribution resembles (3.11):

$$\pi(\underline{y}, \underline{n}) = \frac{\mathbf{1}(|\underline{y}| = M, |\underline{n}| = N)}{\Xi_{N,\Lambda,M}} \prod_{j \in \Lambda} [\bar{\gamma}_j(n_j)]^{y_j}, \quad \Xi_{N,\Lambda,M} = \sum_{\substack{\underline{y}=(y_s) \in \{0,1\}^\Lambda, \\ \underline{n}=(n_s) \in \mathbb{Z}_+^\Lambda: \\ |\underline{y}|=M, |\underline{n}|=N}} \prod_{l \in \Lambda} [\bar{\gamma}_l(n_l)]^{y_l}. \quad (4.17)$$

**Theorem 4.3.** *Given integer  $M, N \geq 1$ , the MP on  $\left\{ (\underline{y}, \underline{n}) \in \{0,1\}^\Lambda \times \mathbb{Z}_+^\Lambda : |\underline{y}| = M, |\underline{n}| = N \right\}$  with generator  $\mathbf{R} = \left( R[(\underline{y}, \underline{n}); (\underline{y}', \underline{n}')] \right)$  as specified in this sub-section is PRR. The SPs  $\pi(\underline{y}, \underline{n})$  are given by (4.17).*

*Proof.* Again by means of suitable DBEs (4.8), with the help of (4.15).  $\square$

## 4.4 An open-closed network

In this version of the model,  $N$ , the number of tasks, is fixed, but the number of DC's varies due to arrivals and exits. A part of transition rates  $R[(\underline{y}, \underline{n}); (\underline{y}', \underline{n}')] comes from (4.1):$

$$\begin{aligned}
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{k \rightarrow l})] &= \beta_{kl}(n_k, n_l; \underline{y}) \mathbf{1}(n_k \geq 1), \quad k \neq l, \quad k, l \notin \underline{y}, \\
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{i \rightarrow j})] &= \epsilon_{ij}(n_i, n_j; \underline{y}) \mathbf{1}(n_i \geq 1), \quad i \neq j, \quad i, j \in \underline{y}, \\
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{j \rightarrow l})] &= \theta_{jl}(n_j, n_l; \underline{y}) \mathbf{1}(n_j \geq 1), \quad j \in \underline{y}, l \notin \underline{y}, \\
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{k \rightarrow j})] &= \theta_{kj}(n_k, n_j; \underline{y}) \gamma_j(n_j) \mathbf{1}(n_k \geq 1), \quad j \in \underline{y}, k \notin \underline{y}, \\
R[(\underline{y}, \underline{n}); (\underline{y} + \underline{e}^{j \rightarrow j'}, \underline{n})] &= [\bar{\gamma}_j(n_j)]^{-1} \tau_{jj'}(\underline{n}; \underline{y}), \quad j \in \underline{y}, j' \notin \underline{y}.
\end{aligned} \tag{4.18}$$

In addition, we use the rates from (4.9). The DBEs read: for  $i, j, j', k, l \in \Lambda$  with  $k \neq l, j \in \underline{y}$ ,

$$\begin{aligned}
\pi(\underline{y}, \underline{n}) R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{k \rightarrow l})] &= \pi(\underline{y}, \underline{n} + \underline{e}^{k \rightarrow l}) R[(\underline{y}, \underline{n} + \underline{e}^{k \rightarrow l}); (\underline{y}, \underline{n})], \quad y_k = y_l, \quad n_k \geq 1, \\
\pi(\underline{y}, \underline{n}) R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{j \rightarrow k})] &= \pi(\underline{y}, \underline{n} + \underline{e}^{j \rightarrow k}) R[(\underline{y}, \underline{n} + \underline{e}^{j \rightarrow k}); (\underline{y}, \underline{n})], \quad k \notin \underline{y}, \quad n_j \geq 1, \\
\pi(\underline{y}, \underline{n}) R[(\underline{y}, \underline{n}); (\underline{y} + \underline{e}^{j \rightarrow j'}, \underline{n})] &= \pi(\underline{y} + \underline{e}^{j \rightarrow j'}, \underline{n}) R[(\underline{y} + \underline{e}^{j \rightarrow j'}, \underline{n}); (\underline{y}, \underline{n})], \quad j' \notin \underline{y}, \\
\pi(\underline{y}, \underline{n}) R[(\underline{y}, \underline{n}); (\underline{y} + \underline{e}^k, \underline{n})] &= \pi(\underline{y} + \underline{e}^k, \underline{n}) R[(\underline{y} + \underline{e}^k, \underline{n}); (\underline{y}, \underline{n})], \quad k \notin \underline{y}.
\end{aligned} \tag{4.19}$$

We now assume conditions (4.11) and (4.15). The SPs and sub-criticality condition read

$$\pi(\underline{y}, \underline{n}) = \frac{\mathbf{1}(|\underline{n}| = N)}{\Xi_{N, \Lambda}} \prod_{q \in \Lambda} \left[ \frac{\xi_q}{\eta_q} \bar{\gamma}_q(n_q) \right]^{y_q}, \quad \Xi_{N, \Lambda} = \sum_{\substack{\underline{y} = (y_s) \in \{0, 1\}^\Lambda, \\ \underline{n} = (n_s) \in \mathbb{Z}_+^\Lambda: |\underline{n}| = N}} \prod_{q \in \Lambda} \left[ \frac{\xi_q}{\eta_q} \bar{\gamma}_q(n_q) \right]^{y_q} < \infty. \tag{4.20}$$

**Theorem 4.4.** *Fix  $N \in \mathbb{Z}_+$ . If  $\Xi_{N, \Lambda} < \infty$ , the MP with generator  $\mathbf{R} = \left( R[(\underline{y}; \underline{n}); (\underline{y}'; \underline{n}')] \right)$  as above is PRR on state space  $\left\{ (\underline{y}, \underline{n}) \in \{0, 1\}^\Lambda \times \mathbb{Z}_+^\Lambda : |\underline{n}| = N \right\}$ . The SPs are given by (4.20).*

*Proof.* Still the DBEs, now from Eqn (4.19).  $\square$

## 5 A zero-range system in a Jackson-type environment

A zero-range modification arises when we allow the DCs to accumulate in sites  $i \in \Lambda$ . Here  $\underline{y} = (y_s, s \in \Lambda) \in \mathbb{Z}_+^\Lambda$ ; we again set  $|\underline{y}| = \sum_{s \in \Lambda} y_s$  and write  $j \in \underline{y}$  when  $y_j \geq 1$  and  $l \notin \underline{y}$  when  $y_l = 0$ .

## 5.1 A closed-open network

In this sub-section,  $M := |\underline{y}|$  is a conserved quantity. The rates are similar to (4.1): for  $i, j, k, l, p \in \Lambda$ ,

$$\begin{aligned}
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^p)] &= \lambda_p(n_p; \underline{y}) [\gamma_p(n_p)]^{y_p}, \quad R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} - \underline{e}^p)] = \mu_p(n_p; \underline{y}) \mathbf{1}(n_p \geq 1), \\
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{k \rightarrow l})] &= \beta_{kl}(n_k, n_l; \underline{y}) \mathbf{1}(n_k \geq 1), \quad k, l \notin \underline{y}, \quad k \neq l \\
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{i \rightarrow j})] &= \epsilon_{ij}(n_i, n_j; \underline{y}) \mathbf{1}(n_i \geq 1), \quad i, j \in \underline{y}, \quad i \neq j \\
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{j \rightarrow l})] &= \theta_{jl}(n_j, n_l; \underline{y}) \mathbf{1}(n_j \geq 1), \quad j \in \underline{y}, \quad l \notin \underline{y}, \\
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{k \rightarrow j})] &= \theta_{kj}(n_k, n_j; \underline{y}) [\gamma_j(n_j)]^{y_j} \mathbf{1}(n_k \geq 1), \quad j \in \underline{y}, \quad k \notin \underline{y}, \\
R[(\underline{y}, \underline{n}); (\underline{y} + \underline{e}^{j \rightarrow j'}, \underline{n})] &= [\bar{\gamma}_j(n_j)]^{-y_j} \tau_{jj'}(\underline{n}; \underline{y}) [\bar{\gamma}_{j'}(n_{j'})]^{-y_{j'}}, \quad j \in \underline{y}, \quad j' \neq j.
\end{aligned} \tag{5.1}$$

Until the end of Section 5, we suppose irreducibility of collections  $B(\underline{n}; \underline{y})$ ,  $\Theta(\underline{n}; \underline{y})$ ,  $E(\underline{n}; \underline{y})$  and  $T(\underline{n}; \underline{y})$ . In sub-Sections 5.1 and 5.2 intensities  $\epsilon_{ij}(n_i, n_j; \underline{y})$  are supposed to obey:  $\forall i \neq j, i, j \in \underline{y}$ ,

$$\frac{\lambda_i(n_i - 1; \underline{y}) [\gamma_i(n_i - 1)]^{y_i}}{\mu_i(n_i; \underline{y})} \epsilon_{il}(n_i, n_j; \underline{y}) = \frac{\lambda_j(n_j; \underline{y}) [\gamma_j(n_j)]^{y_j}}{\mu_j(n_j + 1; \underline{y})} \epsilon_{ji}(n_j + 1, n_i - 1; \underline{y}), \quad n_i \geq 1, \tag{5.2}$$

which replaces (4.4). Conditions (4.2)–(4.3) remain in place in both sub-Sections 5.1 and 5.2.

The present model gives rise to SPs

$$\pi(\underline{y}, \underline{n}) = \frac{\mathbf{1}(|\underline{y}| = M)}{\Xi_{\Lambda, M}} \prod_{p \in \Lambda} \frac{\bar{\lambda}_p(n_p; \underline{y})}{\bar{\mu}_p(n_p; \underline{y})} \prod_{q \in \Lambda} [\bar{\gamma}_q(n_q)]^{y_q}, \quad \Xi_{\Lambda, M} = \sum_{\substack{\underline{y} = (\underline{y}_s) \in \mathbb{Z}_+^\Lambda: r: y_r \geq 1 \\ |\underline{y}| = M}} \prod_{r: y_r \geq 1} C_r(\underline{y}) \prod_{l: y_l = 0} U_l(\underline{y}), \tag{5.3}$$

and

$$U_l(\underline{y}) = \sum_{n \in \mathbb{Z}_+} \frac{\bar{\lambda}_l(n_l; \underline{y})}{\bar{\mu}_l(n; \underline{y})}, \quad C_r(\underline{y}) = \sum_{n \geq 0} \frac{\bar{\lambda}_r(n; \underline{y})}{\bar{\mu}_r(n; \underline{y})} \bar{\gamma}_r(n)^{y_r}. \tag{5.4}$$

The sub-criticality conditions read:  $\forall \underline{y} \in \mathbb{Z}_+^\Lambda$  with  $|\underline{y}| = M$ ,

$$U_l(\underline{y}) < +\infty, \quad C_r(\underline{y}) < +\infty, \quad \forall l, r \in \Lambda \text{ with } y_l = 0 \text{ and } y_r \geq 1. \tag{5.5}$$

**Theorem 5.1.** *Under conditions (5.5), the MP on  $\{(\underline{y}, \underline{n}) \in \mathbb{Z}_+^\Lambda \times \mathbb{Z}_+^\Lambda : |\underline{y}| = M\}$  with generator  $\mathbf{R} = (R[(\underline{y}, \underline{n}); (\underline{y}', \underline{n}')] )$  as in (5.1) is PRR. The SPs  $\pi(\underline{y}, \underline{n})$  are given by (5.3), (5.4).*

*Proof.* Still the DBEs, now for rates (5.1). The DBEs are as in (4.8), and after cancellations are reduced to (4.2), (4.3) and (5.2).  $\square$

## 5.2 An open-open network

In this model we allow both the tasks and DCs to come and leave. Correspondingly, the rates (5.1) are complemented in a manner similar to (4.9):

$$R[(\underline{y}, \underline{n}); (\underline{y} + \underline{e}^p, \underline{n})] = \xi_p \bar{\gamma}_p(n_p), \quad R[(\underline{y}, \underline{n}); (\underline{y} - \underline{e}^p, \underline{n})] = \eta_p \mathbf{1}(y_p \geq 1). \quad (5.6)$$

As in sub-Section 4.2, we assume (4.2) and replace (4.3) by (4.10). We also assume (4.11). (In (4.10),  $\underline{y} \in \{0, 1\}^\Lambda$  is replaced with  $\underline{y} \in \mathbb{Z}_+^\Lambda$ , and in (4.11) the restriction  $j' \notin \underline{y}$  removed.) As in sub-Section 5.1, condition (5.2) is also in place.

Assuming  $U_l(\underline{y})$  and  $C_l(\underline{y})$  as in (5.4), the SPs and sub-criticality condition now read

$$\pi(\underline{y}, \underline{n}) = \Xi_\Lambda^{-1} \prod_{p \in \Lambda} \frac{\bar{\lambda}_p(n_p; \underline{y})}{\bar{\mu}_p(n_p; \underline{y})} \prod_{q \in \Lambda} \left[ \frac{\xi_q}{\eta_q} \bar{\gamma}_q(n_q) \right]^{y_q}, \quad \Xi_\Lambda = \sum_{\underline{y}=(y_s) \in \mathbb{Z}_+^\Lambda} \prod_{r: y_r \geq 1} C_r(\underline{y}) \prod_{l: y_l=0} U_l(\underline{y}) < \infty. \quad (5.7)$$

**Theorem 5.2.** *If  $\Xi_\Lambda < +\infty$ , the MP on  $\mathbb{Z}_+^\Lambda \times \mathbb{Z}_+^\Lambda$  with generator  $\mathbf{R} = \left( R[(\underline{y}, \underline{n}); (\underline{y}', \underline{n}')] \right)$  as in (5.1), (5.6) is PRR. The SPs  $\pi(\underline{y}, \underline{n})$  are given by (5.7).*

*Proof.* The DBEs again. The added equations (5.6) are treated similarly to (4.14).  $\square$

## 5.3 A closed-closed network

Let us now suppose that both  $|\underline{y}|$  and  $|\underline{n}|$  are fixed:  $|\underline{y}| = M$  and  $|\underline{n}| = N$ . The rates are as in Eqn (5.1), with top two lines discarded. In this sub-section, we assume conditions (4.15) and (4.16), with specification  $\underline{y} \in \{0, 1\}$  replaced by  $\underline{y} \in \mathbb{Z}_+^\Lambda$  and condition  $j' \notin \underline{y}$  removed. The only exception is that the bottom line in (4.15) is now replaced with

$$\epsilon_{ij}(n_i, n_j; \underline{y}) [\gamma_i(n_i - 1)]^{y_i} = [\gamma_j(n_j)]^{y_j} \epsilon_{ji}(n_j + 1, n_i - 1; \underline{y}), \quad i \neq j, \quad i, j \in \underline{y}, \quad n_i \geq 1. \quad (5.8)$$

The SP distribution mimicks (4.17):

$$\pi(\underline{y}, \underline{n}) = \frac{\mathbf{1}(|\underline{y}| = M, |\underline{n}| = N)}{\Xi_{N, \Lambda, M}} \prod_{j \in \Lambda} [\bar{\gamma}_j(n_j)]^{y_j}, \quad \Xi_{N, \Lambda, M} = \sum_{\substack{\underline{y}=(y_s), \underline{n}=(n_s) \in \mathbb{Z}_+^\Lambda: \\ |\underline{y}|=M, |\underline{n}|=N}} \prod_{l \in \Lambda} [\bar{\gamma}_l(n_l)]^{y_l}. \quad (5.9)$$

The above DBEs and symmetry conditions (including (5.8)) lead to Theorem 5.3:

**Theorem 5.3.** *The MP on state space  $\left\{ (\underline{y}, \underline{n}) \in \mathbb{Z}_+^\Lambda \times \mathbb{Z}_+^\Lambda : |\underline{y}| = M, |\underline{n}| = N \right\}$  with generator  $\mathbf{R} = \left( R[(\underline{y}, \underline{n}); (\underline{y}', \underline{n}')] \right)$  as in (5.1) is PRR. The SPs  $\pi(\underline{y}, \underline{n})$  are given by (5.9).*

## 5.4 An open-closed network

Here – as in sub-Section 4.4 – we only fix  $N$ . The rates follow (4.18) and consists of

$$\begin{aligned}
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{k \rightarrow l})] &= \beta_{kl}(n_k, n_l; \underline{y}) \mathbf{1}(n_k \geq 1), \quad k \neq l, \quad k, l \notin \underline{y}, \\
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{i \rightarrow j})] &= \epsilon_{ij}(n_i, n_j; \underline{y}) \mathbf{1}(n_i \geq 1), \quad i \neq j, \quad i, j \in \underline{y}, \\
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{j \rightarrow l})] &= \theta_{jl}(n_j, n_l; \underline{y}) \mathbf{1}(n_j \geq 1), \quad j \in \underline{y}, \quad l \notin \underline{y}, \\
R[(\underline{y}, \underline{n}); (\underline{y}, \underline{n} + \underline{e}^{k \rightarrow j})] &= \theta_{kj}(n_k, n_j; \underline{y}) [\gamma_j(n_j)]^{y_j} \mathbf{1}(n_k \geq 1), \quad j \in \underline{y}, \quad k \notin \underline{y}, \\
R[(\underline{y}, \underline{n}); (\underline{y} + \underline{e}^{j \rightarrow j'}, \underline{n})] &= [\bar{\gamma}_j(n_j)]^{-y_j} \tau_{jj'}(\underline{n}; \underline{y}) [\bar{\gamma}_{j'}(n_{j'})]^{-y_{j'}}, \quad j \neq j', \quad j \in \underline{y},
\end{aligned} \tag{5.10}$$

plus rates from Eqns (5.6). As in sub-Section 4.4, we now assume conditions (4.11) and (4.15), modified like above (including (5.9)). The SPs and sub-criticality condition read

$$\pi(\underline{y}, \underline{n}) = \frac{\mathbf{1}(|\underline{n}| = N)}{\Xi_{N, \Lambda}} \prod_{l \in \Lambda} \left[ \frac{\xi_l}{\eta_l} \bar{\gamma}_l(n_l) \right]^{y_l}, \quad \Xi_{N, \Lambda} = \sum_{\substack{\underline{y}=(y_s) \underline{n}=(n_s) \in \mathbb{Z}_+^\Lambda: \\ |\underline{n}|=N}} \prod_{l \in \Lambda} \left[ \frac{\xi_l}{\eta_l} \bar{\gamma}_l(n_l) \right]^{y_l} < \infty. \tag{5.11}$$

The DBEs in this case yield

**Theorem 5.4.** *Fix  $N \in \mathbb{Z}_+$  and consider the MP on  $\{(\underline{y}, \underline{n}) \in \mathbb{Z}_+^\Lambda \times \mathbb{Z}_+^\Lambda : |\underline{n}| = N\}$  with generator  $\mathbf{R} = \left( R[(\underline{y}, \underline{n}); (\underline{y}', \underline{n}')] \right)$  as in (5.10). If  $\Xi_{N, \Lambda} < \infty$ , Assuming (5.11), it is PRR. The SPs are given by (5.11).*

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